"श्री गणेशाय नमः"

Date: 23/11/22

UNIT - II

"Partial Differentiation"

① Euler's theoram ② Error & approximation
③ Partial differention. → 12 marks

Ques. If $z(x+y) = x^2 + y^2$, show that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

Here ⇒ $z = \dfrac{x^2+y^2}{(x+y)}$

$$\frac{\partial z}{\partial x} = \frac{\frac{\partial (x^2+y^2)}{\partial x}(x+y) - \frac{\partial (x+y)}{\partial x}(x^2+y^2)}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} = \frac{2x(x+y) - (x^2+y^2)}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} = \frac{2x^2 + 2xy - x^2 - y^2}{(x+y)^2}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{x^2 - y^2 + 2xy}{(x+y)^2}}$$

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$$z = \frac{x^2+y^2}{(x+y)}$$

$$\frac{\partial z}{\partial y} = \frac{\frac{d(x^2+y^2)}{dy}(x+y) - \frac{d(x+y)}{dy}(x^2+y^2)}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{2y(x+y) - (x^2+y^2)}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{2xy+2y^2-x^2-y^2}{(x+y)^2}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{y^2-x^2+2xy}{(x+y)^2}}$$

$$z = \frac{x^2+y^2}{(x+y)}$$

$$\left[\left(\frac{\partial z}{\partial x}\right) - \frac{\partial z}{\partial y}\right]^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

$$\left[\frac{x^2-y^2+2xy}{(x+y)^2} - \frac{(y^2-x^2+2xy)}{(x+y)^2}\right]^2$$

$$= \underbrace{4\left(1 - \left[\frac{x^2-y^2+2xy}{(x+y)^2}\right] - \left(\frac{y^2-x^2+2xy}{(x+y)^2}\right)\right)}_{R.H.S}$$

$$\left[\frac{x^2-y^2\cancel{+2xy}-y^2+x^2\cancel{-2xy}}{(x+y)^2}\right]^2 =$$

$$\left[\frac{2x^2-2y^2}{(x+y)^2}\right]^2 \Rightarrow \left[\frac{2(x^2-y^2)}{(x+y)^2}\right]^2 = \frac{4\cancel{(x+y)^2}(x-y)^2}{\cancel{(x+y)^2}(x+y)^2}$$

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$$= \frac{4(x-y)^2}{(x+y)^2} \Leftarrow L.H.S$$

Now R.H.S →

$$4\left(1-\left(\frac{dz}{dx}+\frac{dz}{dy}\right)\right] = 4\left[1-\frac{4xy}{(x+y)^2}\right]$$

$$= 4\left[\frac{(x+y)^2-4xy}{(x+y)^2}\right] = \frac{4(x-y)^2}{(x+y)^2}$$

R.H.S

$$\boxed{L.H.S = R.H.S}$$ Hence Proved.

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Que. 2 If $u = (x^2+y^2+z^2)^{-1/2}$, $x^2+y^2+z^2 \neq 0$

Prove that

(1) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u$

Assignment (2) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 z}{\partial z^2} = 0$

$$\frac{\partial u}{\partial x} = -\frac{\cancel{2}x}{\cancel{2}}(x^2+y^2+z^2)^{-3/2} = -\frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial u}{\partial y} = -\frac{\cancel{2}y}{\cancel{2}}(x^2+y^2+z^2)^{-3/2} = -\frac{y}{(x^2+y^2+z^2)^{3/2}}$$

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$$\frac{\partial u}{\partial z} = \frac{-z}{(x^2+y^2+z^2)^{3/2}}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{-1}{(x^2+y^2+z^2)^{3/2}}(x^2+y^2+z^2)$$

$$= -(x^2+y^2+z^2)^{-1/2} = -u$$

Hence Proved.

Assignment Que. 2 If $u = e^{xyz}$, Prove that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1+3xyz+x^2y^2z^2)e^{xyz}$

Assignment Que. 3 If $u = \log(x^3+y^3+z^3-3xyz)$

Prove That (i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$.

(ii) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 \cdot u = \frac{-9}{(x+y+z)^2}$.

Que. 4 Assignment If $x^x y^y z^z = C$ then. Prove that

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}.$$

$$\log(x^x y^y z^z) = \log C$$

$$x\log x + y\log y + z\log z = \log C$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial z}{\partial y} \Rightarrow \frac{\partial}{\partial y}\left(x\log x + y\log y + z\log z\right) = \frac{\partial (\log C}{\partial y}$$

$$\frac{\partial (x\log x)}{\partial y} + \frac{\partial (y\log y)}{\partial y} + \frac{\partial (z\log z)}{\partial y} = 0$$

$$0 + \left(\log y + y \cdot \frac{1}{y}\right) + \frac{\partial z}{\partial y} \cdot \log z + \frac{\partial (\log z)}{\partial y} \cdot z = 0$$

$$\log y + y \cdot \frac{1}{y} + \frac{\partial z}{\partial z} \cdot \frac{dz}{\partial y} \cdot \log z + \frac{\partial (\log z)}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot z = 0$$

$$\log y + y \cdot \frac{1}{y} + \log z \cdot \frac{\partial z}{\partial y} + z \cdot \frac{1}{z} \cdot \frac{dz}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{[1 + \log y]}{[1 + \log z]}$$

$$\frac{\partial z}{\partial x} = -\frac{[1 + \log x]}{[1 + \log z]}$$

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★ Euler's theoram –

If $f(x,y)$ is a homogenous func^n of x,y is ~~f(x,y)~~ of degree n then,

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$$

Relation b/w second order derivative.

$$x^2\frac{\partial^2 f}{\partial x^2}+xy\frac{\partial^2 f}{\partial x\,\partial y}=(n-1)\frac{\partial f}{\partial x}$$

$$xy\frac{\partial^2 f}{\partial y\,\partial x}+y^2\frac{\partial^2 f}{\partial y^2}=(n-1)\frac{\partial f}{\partial y}$$

$$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\,\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=n(n-1)f$$

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Que. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u$

Assignment

$= 2\cos 3u \sin u$

$$\frac{\partial y}{\partial x} = f\left(\frac{y}{x}\right)$$

$$\tan u = \frac{x^3+y^3}{x-y} = \frac{x^{\cancel{3}2}}{\cancel{x}}\left[\frac{1+(y/x)^3}{1-(y/x)}\right]$$

$$\boxed{n=2}$$

Let $f = \tan u$

→ By Euler's theoram,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$x\frac{\partial(\tan u)}{\partial x} + y\frac{\partial(\tan u)}{\partial y} = 2\tan u$$

$u = f\left(\frac{y}{x}\right) \quad x\left(\frac{du}{dx}\right) + y\left(\frac{du}{dy}\right)$

$$x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2 \tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u \cdot \cos^2 u$$
$$= 2 \sin u \cos u$$
$$= \underline{\sin 2u} \text{ Ans.}$$

in2u

Assign. Que. 1 If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$

Assign. Que. 2 If $u = \frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$, verify the Euler's theoram

Ans-1) $\sin u = \frac{x^2+y^2}{(x+y)} = \frac{x^2}{x}\left(\frac{1+\left(\frac{y}{x}\right)^2}{1+\frac{y}{x}}\right) \quad \boxed{n=1}$

$f = \sin u$

By Euler's theoram

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$x \cos u \frac{du}{dx} + y \cos u \frac{du}{dy} = \sin u$$

$$x\frac{du}{dx} + y\frac{du}{dy} = \frac{\sin u}{\cos u} = \underline{\tan u}$$

Hence Proved!

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→ Only one que. in Exam (1, or 2, or 3 marks)

★ Errors and approxiation. —

Let $y = f(x)$, if x increases by δx and coorresponding increases in y is δy then,

$$\text{Let } y = f(x)$$

$$y + \delta y = f(x + \delta x)$$

$$\boxed{\delta y = f(x + \delta x) - f(x)} \quad \text{--- (1)}$$

We know,

$$\frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\delta y = f'(x) \cdot \delta x \quad (\text{Approx.})$$

$$\boxed{\frac{\delta y}{y} = \frac{f'(x)}{f(x)} \cdot \delta x} \quad \text{--- (2)}$$

Thus, if δx is small error then the corresponding error δy in y can bue cal by for. no. (2) the error in x is called absolute. The ratio $\frac{\delta x}{x}$ is relative error and $\frac{\delta x}{x} \times 100 = \% x$.

$f = f(x, y)$

$$\boxed{\delta f = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y} \quad (\text{appro

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Que. Evalute $\sqrt{99}$ approximately.

$\Rightarrow$ Let, $y = f(x) = \sqrt{x}$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$\frac{f(x+\delta x) - f(x)}{\delta x} = f'(x)$$

$$f(x+\delta x) - f(x) = f'(x) \cdot \delta x$$

Now writing, $99 = 100 - 1$

Take, $x = 100$, $\delta x = -1$

$$f(100-1) - \sqrt{x}) = \frac{1}{2\sqrt{x}} \cdot (-1)$$

$$f(99) - \sqrt{x} = \frac{-1}{2\sqrt{x}}$$

$$f(99) = 10 - \frac{1}{20} = \frac{200-1}{20} = \frac{199}{20}$$

$$\boxed{f(99) = 9.95}$$

Que Assign. Evaluate cube root of 127 approximately.

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Que. Find the percentage error in the area of the rectangle when an error of +1 % made in measure its length & breath.

Sol.^{n} => Let A = Area
x and y are length breath

$$A = xy$$

$$\log A = \log x + \log y$$

$$\frac{\delta A}{A} = \frac{\delta x}{x} + \frac{\delta y}{y}$$

$$\frac{\delta A}{A} \times 100 = \frac{\delta x}{x} \times 100 + \frac{\delta y}{y} \times 100$$

$$A\,\% = 1 + 1 = 2\,\%$$ Ans.

Que. The $T = 2\pi\sqrt{\frac{l}{g}}$ of a simple pendullam

Discuss the maxima & minima

<u>Que.</u> $f(x,y) = x^3y^2(1-x-y)$

<u>Sol.^{n}</u> $\Rightarrow$ $f = x^3y^2 - x^4y^2 - x^3y^3$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$$

$$\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2 = 0$$

$x^2y^2(3-4x-3y)=0$ | $x^3y(2-2x-3y)=0$

$3-4x-3y=0$ | $2-2x-3y=0$

$\times(-1)$ $4x+3y-3=0$ — ① | $-2x-3y+2=0$ — ②

from eq.^{n} ① & ②

$$4x+\cancel{3y}-3=0 \quad \text{—①}$$
$$-2x-\cancel{3y}+2=0 \quad \text{—②}$$

$$2x-1=0$$

$$\boxed{x=\frac{1}{2}}$$

Put in eq. ①

$$\cancel{4}^{2}\left(\frac{1}{\cancel{2}}\right)+3y-3=0$$

$$3y-1=0$$

$$\boxed{y=\frac{1}{3}}$$

$-3x^2y^3$

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$x = \frac{1}{2}$, $y = \frac{1}{3}$

$\frac{df}{dx} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$

$\frac{df}{dy} = 2x^3y - 2x^4y - 3x^3y^2$

$$r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)} = \left(6xy^2 - 12x^2y^2 - 6xy^3\right)_{\left(\frac{1}{2},\frac{1}{3}\right)}$$

$$r = \left(\cancel{6}^{3} \times \frac{1}{\cancel{2}} \times \frac{1}{\cancel{9}_{3}}\right) - \left(\cancel{12}^{3} \times \frac{1}{\cancel{4}} \times \frac{1}{\cancel{9}_{3}}\right) - \left(\cancel{6}^{3} \times \frac{1}{\cancel{2}} \times \frac{1}{\cancel{27}_{9}}\right)$$

$$r = \cancel{\frac{1}{3}} - \cancel{\frac{1}{3}} - \frac{1}{9} \Rightarrow \boxed{r = -\frac{1}{9}}$$

$$t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)} = \left(2x^3 - 2x^4 - 6x^3y\right)_{\left(\frac{1}{2},\frac{1}{3}\right)}$$

$$t = 2\left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{2}\right)^4 - \cancel{6}^{2}\left(\frac{1}{2}\right)^3\left(\frac{1}{\cancel{3}}\right)$$

$$t = \cancel{\frac{2}{8}} - \frac{\cancel{2}}{\cancel{16}}\frac{1}{8} - \cancel{\frac{2}{8}} \Rightarrow \boxed{t = -\frac{1}{8}}$$

$$S = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial\left(2x^3 - 2x^4 - 6x^3y\right)}{\partial x} =$$

$$S = \left(6x^2 - 8x^3 - 18x^2y\right)_{\left(\frac{1}{2},\frac{1}{3}\right)}$$

$$S = \cancel{6}^{3} \times \frac{1}{\cancel{4}_{2}} - 8 \times \frac{1}{8} - \cancel{18}^{\cancel{6}\,3} \times \frac{1}{\cancel{4}_{2}} \times \frac{1}{\cancel{3}} = \cancel{\frac{3}{2}} - \frac{8}{27} - \cancel{\frac{3}{2}}$$

$$\boxed{S = \frac{-8}{27}}$$